

Generalized Calabi-Yau metric and Generalized Monge-Ampère equation

Chris M. Hull^a, Ulf Lindström^b, Martin Roček^c,
Rikard von Unge^d and Maxim Zabzine^b

^a *The Blackett Laboratory, Imperial College London
Prince Consort Road, London SW7 2AZ, U.K.*

^b *Theoretical Physics, Department of Physics and Astronomy,
Uppsala University, Box 516, SE-751 20 Uppsala, Sweden*

^c *C.N.Yang Institute for Theoretical Physics, Stony Brook University,
Stony Brook, NY 11794-3840, USA*

^d *Institute for Theoretical Physics, Masaryk University,
611 37 Brno, Czech Republic*

Abstract

In the neighborhood of a regular point, generalized Kähler geometry admits a description in terms of a single real function, the generalized Kähler potential. We study the local conditions for a generalized Kähler manifold to be a generalized Calabi-Yau manifold and we derive a non-linear PDE that the generalized Kähler potential has to satisfy for this to be true. This non-linear PDE can be understood as a generalization of the complex Monge-Ampère equation and its solutions give supergravity solutions with metric, dilaton and H -field.

1 Introduction

Generalized geometry, initiated by N. Hitchin in [1] has attracted considerable interest in both physics and mathematics. The essential idea behind generalized geometry is to replace the tangent bundle T by $T \oplus T^*$, the tangent plus the cotangent bundle. This replacement leads to many new and interesting geometrical concepts which can play a prominent role in string theory. The subject was further developed by Hitchin's students [2, 3, 4].

In this paper we study the conditions for a generalized Kähler manifold to admit a generalized Calabi-Yau metric structure. Our analysis is local and we do not consider global issues. We take as a starting point our previous results [5] on the local description of generalized Kähler manifolds in terms of a single function, the generalized Kähler potential. We derive a non-linear PDE for this potential which comes from the generalized Calabi-Yau conditions. We will refer to this non-linear PDE as the generalized Monge-Ampère equation, since it generalizes the complex Monge-Ampère equation which appears in the local description of the standard Calabi-Yau manifolds. This idea has previously been discussed in [6] and some partial results were presented there. Here we give a full derivation for the general case.

On the physics side we relate the Type II supergravity solutions for metric, dilaton and NS-flux to the world-sheet description in terms of the $N = (2, 2)$ non-linear sigma model. The generalized Calabi-Yau conditions appear naturally in supergravity solutions, [7], [8] and [9] (see [10] for a nice summary and further references). On the other hand, generalized Kähler manifolds [2] of any complex dimension appear as target spaces for sigma models with $N = (2, 2)$ supersymmetry [11] and classically any generalized Kähler potential K (satisfying some minor conditions required for the geometry to be non degenerate) gives a generalized Kähler manifold. The proposed generalized Monge-Ampère equation appears at the quantum level when one requires conformality of the sigma model at the one loop level. The connection to standard 10 dimensional supergravity solutions come when one also imposes the correct value of the central charge.

The paper is organized as follows: In Section 2 we review the basic notions of generalized complex, generalized Kähler and generalized Calabi-Yau geometries. Then, in Section 3, using the local description of generalized Kähler geometry, we present explicit formulas for the pure spinors that encode the geometry in terms of the generalized Kähler potential. We use the fact that the generalized Calabi-Yau metric structure is equivalent

to the existence of supersymmetric supergravity solutions¹. We show that the compatibility conditions of the pure spinors are trivially satisfied except for the normalization condition which gives a non-linear PDE (the generalized Monge-Ampère equation) for the generalized Kähler potential. Section 4 deals with a few special cases of the generalized Monge-Ampère equation and some simple solutions. In Section 5 we elaborate on the geometrical aspects of the generalized Monge-Ampère equation. In Section 6 we comment on the relation of our results to the β -function calculations for supersymmetric sigma model. Section 7 gives the summary and discusses some open questions.

2 Generalized Kähler and Calabi-Yau geometries

In this section we briefly review the relevant concepts in generalized geometry. Since our analysis is local we ignore complications related to global issues². This section does not contain original material. For further details the reader may consult [2, 10, 12].

Consider a smooth manifold M . The tangent plus cotangent bundle $T \oplus T^*$ has a natural inner product

$$\langle X + \eta, Y + \xi \rangle = \frac{1}{2}(i_X \xi + i_Y \eta) , \quad (2.1)$$

where X, Y are vector fields and η, ξ are one-forms. The Courant bracket on $T \oplus T^*$ is

$$[X + \eta, Y + \xi]_c = \{X, Y\} + \mathcal{L}_X \xi - \mathcal{L}_Y \eta - \frac{1}{2}d(i_X \xi - i_Y \eta) , \quad (2.2)$$

and is a generalization of the Lie bracket naturally defined on T . A *generalized complex structure* is defined as a decomposition $(T \oplus T^*) \otimes \mathbb{C} = L + \bar{L}$ where L is a maximally isotropic sub-bundle involutive with respect to the Courant bracket and \bar{L} is its complex conjugate. This decomposition can be encoded in terms of an endomorphism \mathcal{J} of $T \oplus T^*$ with some additional properties, such that L is the $+i$ -eigenbundle of \mathcal{J} .

A *generalized Kähler structure* is defined as two commuting generalized complex structures \mathcal{J}_1 and \mathcal{J}_2 such that the quadratic form $\langle \mathcal{J}_1 \mathcal{J}_2(X + \xi), (X + \xi) \rangle$ is positive definite. Equivalently the generalized Kähler structure gives rise to the decomposition $(T \oplus T^*) \otimes \mathbb{C} = L_1^+ \oplus L_1^- \oplus \bar{L}_1^+ \oplus \bar{L}_1^-$ where L_1^+ is the $+i$ -eigenbundle for both of $\mathcal{J}_{1,2}$ and L_1^- is the

¹That is, solutions that preserve some of the supersymmetry of the target.

²Concretely, we work in a contractible coordinate patch and assume that $H = dB$ is exact in that patch, so that the effect of H can be encoded into the appropriate B-transform. We do not need to use the twisted Courant bracket and the twisted de Rham differential here, and we do not discuss global issues.

$+i$ -eigenbundle for \mathcal{J}_1 and $-i$ - eigenbundle for \mathcal{J}_2 . Due to Gualtieri's theorem [2], generalized Kähler geometry is equivalent to bihermitian Gates-Hull-Roček geometry [11] which appears as the target space geometry for $N = (2, 2)$ supersymmetric sigma models. We review the details of Gates-Hull-Roček geometry and the Gualtieri map in the appendices.

Differential forms play a crucial role in generalized geometry. We may regard the differential forms on M as spinors for the bundle $T \oplus T^*$. A section $X + \xi$ of $T + T^*$ acts on a form ρ according to

$$(X + \xi) \cdot \rho = i_X \rho + \xi \wedge \rho \quad (2.3)$$

and this satisfies the Clifford algebra identity for the indefinite metric (2.1) on $T \oplus T^*$. An invariant bilinear form on spinors (differential forms) is given by

$$(\rho_1, \rho_2) = \sum_j (-1)^j [\rho_1^{2j} \wedge \rho_2^{d-2j} + \rho_1^{2j+1} \wedge \rho_2^{d-2j-1}] , \quad (2.4)$$

where $\rho_1 = \sum \rho_1^i$ and $\rho_2 = \sum \rho_2^i$ are linear combinations forms of different degrees, the upper index indicating the form degree. This is called the Mukai pairing of forms. The spinor is called pure if it annihilates a maximal isotropic subspace of $T \oplus T^*$ (or its complexification $(T \oplus T^*) \otimes \mathbb{C}$).

A *generalized Calabi-Yau structure* is defined as a closed pure spinor ρ such that $(\rho, \bar{\rho}) \neq 0$ everywhere on M , where $\bar{\rho}$ is the complex conjugate of ρ . A closed pure spinor ρ gives rise to a decomposition $(T \oplus T^*) \otimes \mathbb{C} = L + \bar{L}$ where L is a maximally isotropic subbundle involutive with respect to the Courant bracket. Therefore a generalized Calabi-Yau manifold is a special case of a generalized complex manifold.

A *generalized Calabi-Yau metric structure*³ is defined as a pair of closed pure spinors ρ_1 and ρ_2 such that the corresponding generalized complex structures \mathcal{J}_1 and \mathcal{J}_2 give rise to a generalized Kähler structure and moreover that $(\rho_1, \bar{\rho}_1) = \alpha(\rho_2, \bar{\rho}_2) \neq 0$ for some non-zero constant α .

The geometric data encoded by the pair of pure spinors defining a generalized Calabi-Yau metric structure may be found by observing that it gives rise to generalized Kähler geometry. Using the Gualtieri map we may then read off the metric $g_{\mu\nu}$ and the closed NS 3-form $H_{\mu\nu\rho}$. Furthermore the dilaton Φ is defined via normalization of pure spinors as follows

$$(\rho_1, \bar{\rho}_1) = \alpha(\rho_2, \bar{\rho}_2) = e^{-2\Phi} \text{vol}_g = e^{-2\Phi} \sqrt{g} \, dx^1 \wedge \dots \wedge dx^D, \quad (2.5)$$

³There is no fully accepted use of term “generalized Calabi-Yau” and different authors use different adjectives like “weak”, “strong” to represent different version of the generalized Calabi-Yau condition. We follow Gualtieri's terminology [2].

where $g = \det(g_{\mu\nu})$. This data $(g_{\mu\nu}, H_{\mu\nu\rho}, \Phi)$ coming from the generalized Calabi-Yau metric structure is a Type II supersymmetric supergravity solution. It automatically solves the equation

$$R_{\mu\nu}^{(+)} + 2\nabla_{\mu}^{(-)}\partial_{\nu}\Phi = 0 \ , \quad (2.6)$$

where $R_{\mu\nu}^{(\pm)}$ is the Ricci tensor for the connection with torsion $\nabla^{(\pm)} = \nabla \pm \frac{1}{2}g^{-1}H$ where ∇ is the Levi-Civita connection. Indeed, the symmetric and antisymmetric parts of (2.6) are the bosonic equations of motion of Type II supergravity with the RR-fields set to zero. For further relevant results and references the reader may consult the nice review [10].

Our strategy will be to take our local description of generalized Kähler geometry in terms of the potential K and construct the corresponding closed pure spinors for two generalized complex structures. Through the theorem in [8, 13] we know that the existence of a pair of compatible closed pure spinors implies the existence of supersymmetric supergravity solutions. Imposing the generalized Calabi-Yau metric conditions (i.e. the compatibility conditions for the pure spinors) we arrive at a non-linear PDE for K . The Gualtieri map plays a crucial role in our construction.

3 Generalized Monge-Ampère equation

Recently, in [5], it was shown that, in the neighborhood of a regular point, generalized Kähler geometry can be encoded in terms of a single real function: the generalized Kähler potential K . The construction roughly goes as follows. We take $\mathbb{C}^{\frac{D}{2}}$ and divide the complex coordinates in four groups $\mathbb{C}^{\frac{D}{2}} = \mathbb{C}^{d_c} + \mathbb{C}^{d_t} + \mathbb{C}^{d_s} + \mathbb{C}^{d_{\bar{s}}}$ with the following notations for the coordinates $\phi = (\phi, \bar{\phi})$, $\chi = (\chi, \bar{\chi})$, $X_L = (\mathbf{X}_l, \bar{\mathbf{X}}_{\bar{l}})$, $X_R = (\mathbf{X}_r, \bar{\mathbf{X}}_{\bar{r}})$ in the respective subspaces. The coordinates are adapted to the complex structures J_{\pm} but in a more involved way than in ordinary Kähler geometry. In particular ϕ, χ and \mathbf{X}_l are holomorphic with respect to J_+ while there is no such simple relation between X_R and J_+ . On the other hand, $\phi, \bar{\chi}$ and \mathbf{X}_r are holomorphic with respect to J_- while there is no such simple relation between X_L and J_- .

Introducing the real function $K(\phi, \chi, X_L, X_R)$ which we call the generalized Kähler potential we derive formulas for (g, J_{\pm}, H) in terms of derivatives of K such that they become the geometric data for a generalized Kähler manifold. In appendix A we review the relevant relations.

In the language of supersymmetric sigma models, the function K corresponds to a Lagrangian density in $N = (2, 2)$ superspace and the coordinates correspond to different

sets of superfields: $(\phi, \bar{\phi})$ for chiral and anti-chiral; $\chi = (\chi, \bar{\chi})$ for twisted chiral and twisted anti-chiral; $X_L = (\mathbf{X}_l, \bar{\mathbf{X}}_{\bar{l}})$ for left semi-chiral and left semi-anti-chiral; $X_R = (\mathbf{X}_r, \bar{\mathbf{X}}_{\bar{r}})$ for right semi-chiral and right semi-anti-chiral. The sigma model provides a useful way of deriving and manipulating the rather complicated expressions for the various geometrical objects. For further details about the sigma model interpretation the reader may consult [5, 14, 15].

It is important to stress that the choice of generalized Kähler potential is not unique. There are many choices that lead to the same geometric objects. This ambiguity in the choice of K can be understood both from geometry and from the sigma model point of view, see [16].

In the following discussion we adopt the following short-hand notations for the derivatives of K : $K_C = \partial_\phi K = (K_c, K_{\bar{c}}) = (\partial_\phi K, \partial_{\bar{\phi}} K)$, $K_T = \partial_\chi K = (K_t, K_{\bar{t}}) = (\partial_\chi K, \partial_{\bar{\chi}} K)$, $K_L = \partial_{X_L} K = (K_l, K_{\bar{l}}) = (\partial_{\mathbf{X}_l} K, \partial_{\bar{\mathbf{X}}_{\bar{l}}} K)$, $K_R = \partial_{X_R} K = (K_r, K_{\bar{r}}) = (\partial_{\mathbf{X}_r} K, \partial_{\bar{\mathbf{X}}_{\bar{r}}} K)$, where we suppress all coordinates indices. Analogously we define the matrices of double derivatives of K , e.g. $K_{l\bar{r}}$ is our notation for the matrix of second derivatives $\partial_{\mathbf{X}_l} \partial_{\bar{\mathbf{X}}_{\bar{r}}} K$ etc. We use matrix notation and suppress all indices.

Next we would like to use K to construct the pure spinors which encode the generalized Kähler geometry. Our ansatz for the pure spinors is

$$\rho_{1,2} = N_{1,2} \wedge e^{R_{1,2} + iS_{1,2}} , \quad (3.7)$$

with the following notation

$$\begin{aligned} N_1 &= e^{f(\phi)} d\phi^1 \wedge \dots \wedge d\phi^{d_c} , \\ N_2 &= e^{g(\chi)} d\chi^1 \wedge \dots \wedge d\chi^{d_t} , \\ R_1 &= -d(K_L dX_L) , \\ R_2 &= -d(K_R dX_R) , \\ S_1 &= d(K_T J d\chi + K_L J dX_L - K_R J dX_R) , \\ S_2 &= -d(K_C J d\phi + K_L J dX_L + K_R J dX_R) , \end{aligned}$$

where d is the de Rham differential and J is the trivial diagonal complex structure so that $K_L J d\phi$ is short-hand for $iK_l dX_l - iK_{\bar{l}} d\bar{X}_{\bar{l}}$ and so on. Here f and g depend only on ϕ and χ respectively and are introduced to take care of the ambiguity in the definition of $N_{1,2}$. Namely, under a change of coordinates $\phi'(\phi)$ and $\chi'(\chi)$ the exponentials $e^{f(\phi)}$ and $e^{g(\chi)}$ transform as densities,

$$e^{f'(\phi')} = e^{f(\phi)} \det \left(\frac{\partial \phi'}{\partial \phi} \right) , \quad e^{g'(\chi')} = e^{g(\chi)} \det \left(\frac{\partial \chi'}{\partial \chi} \right) . \quad (3.8)$$

so that $N_{1,2}$ themselves are invariant.

First of all one has to check that these are the pure spinors which annihilate the correct subspace of $(T \oplus T^*) \times \mathbb{C}$, i.e. those subspace defined by the generalized complex structures \mathcal{J}_1 and \mathcal{J}_2 . The proof of this statement involves the Gualtieri map and is technical. We give the details in appendix B. The pure spinors are trivially closed $d\rho_{1,2} = 0$ and moreover $(\rho_1, \rho_2) = (\rho_1, \bar{\rho}_2) = 0$ (see appendix B). The only remaining condition for generalized Calabi-Yau metric is $(\rho_1, \bar{\rho}_1) = \alpha(\rho_2, \bar{\rho}_2)$ for some positive constant α . Explicitly, this condition becomes

$$\begin{aligned}
& (-1)^{\frac{d_c(d_c-1)}{2}} e^{f(\phi)} e^{\bar{f}(\bar{\phi})} \text{Pf} \begin{pmatrix} 0 & -K_{l\bar{l}} & -K_{lr} & 0 & 0 & -K_{l\bar{t}} \\ K_{l\bar{l}} & 0 & 0 & K_{l\bar{r}} & K_{l\bar{t}} & 0 \\ K_{rl} & 0 & 0 & K_{r\bar{r}} & K_{rt} & 0 \\ 0 & -K_{\bar{r}l} & -K_{\bar{r}r} & 0 & 0 & -K_{\bar{r}\bar{t}} \\ 0 & -K_{t\bar{l}} & -K_{tr} & 0 & 0 & -K_{t\bar{t}} \\ K_{t\bar{l}} & 0 & 0 & K_{t\bar{r}} & K_{t\bar{t}} & 0 \end{pmatrix} \\
& = (-1)^{\frac{d_t(d_t-1)}{2}} \alpha e^{g(\chi)} e^{\bar{g}(\bar{\chi})} \text{Pf} \begin{pmatrix} 0 & K_{l\bar{l}} & 0 & K_{l\bar{r}} & 0 & K_{l\bar{c}} \\ -K_{l\bar{l}} & 0 & -K_{l\bar{r}} & 0 & -K_{l\bar{c}} & 0 \\ 0 & K_{r\bar{l}} & 0 & K_{r\bar{r}} & 0 & K_{r\bar{c}} \\ -K_{\bar{r}l} & 0 & -K_{\bar{r}r} & 0 & -K_{\bar{r}c} & 0 \\ 0 & K_{c\bar{l}} & 0 & K_{c\bar{r}} & 0 & K_{c\bar{c}} \\ -K_{\bar{c}l} & 0 & -K_{\bar{c}r} & 0 & -K_{\bar{c}c} & 0 \end{pmatrix}
\end{aligned} \tag{3.9}$$

where we used the definition (2.4) of the Mukai pairing and the usual definition the Pfaffian (see (3.11) below).

For a $2n \times 2n$ antisymmetric matrix $\mathcal{A} = \{a_{\mu\nu}\}$ one associates a two-form

$$a = \sum_{\mu < \nu} a_{\mu\nu} e^\mu \wedge e^\nu \tag{3.10}$$

with the standard basis $\{e^1, e^2, \dots, e^{2n}\}$ in \mathbb{R}^{2n} . The Pfaffian is defined as

$$\frac{1}{n!} a^n = \text{Pf}(\mathcal{A}) e^1 \wedge e^2 \wedge \dots \wedge e^{2n}, \tag{3.11}$$

where a^n denotes the wedge product of n copies of a with itself.

Our conventions are such that α in (2.5) is positive, assuming a Riemannian metric $g_{\mu\nu}$. If $d_c \neq d_t$ then the constant α can be easily absorbed by a rescaling of K , thus giving the same geometry (a rescaling of the generalized Kähler potential will just rescale all geometric objects). However, in the case $d_c = d_t$ (including the case $d_c = d_t = 0$) this is

not possible. Instead, if $d_c = d_t > 0$ we may rescale the ϕ or χ coordinates to get $\alpha = 1$. Only in the case $d_c = d_t = 0$ this is not possible and we get a family of PDEs parametrized by α .

There are two useful relations for the Pfaffian we can use to simplify further the condition (3.9). For $2n \times 2n$ antisymmetric matrix \mathcal{A} and an arbitrary $2n \times 2n$ matrix V we have the relation

$$\text{Pf}(V\mathcal{A}V^T) = \det(V) \text{Pf}(\mathcal{A}) . \quad (3.12)$$

For an arbitrary $n \times n$ -matrix M we may write

$$\text{Pf} \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} = (-1)^{n(n-1)/2} \det M . \quad (3.13)$$

Using the relation (3.12) we can reorder rows and columns by choosing an appropriate V and arrive at a block off-diagonal form. To this we can apply (3.13) arriving at

$$(-1)^{d_s d_c} e^{f(\phi)} e^{\bar{f}(\bar{\phi})} \det \begin{pmatrix} -K_{l\bar{l}} & -K_{lr} & -K_{l\bar{t}} \\ -K_{\bar{r}\bar{l}} & -K_{\bar{r}r} & -K_{\bar{r}\bar{t}} \\ -K_{t\bar{l}} & -K_{tr} & -K_{t\bar{t}} \end{pmatrix} = \alpha e^{g(\mathbf{x})} e^{\bar{g}(\bar{\mathbf{x}})} \det \begin{pmatrix} K_{l\bar{r}} & K_{l\bar{l}} & K_{l\bar{c}} \\ K_{r\bar{r}} & K_{r\bar{l}} & K_{r\bar{c}} \\ K_{c\bar{r}} & K_{c\bar{l}} & K_{c\bar{c}} \end{pmatrix} \quad (3.14)$$

where we have redefined the proportionality constant α (which is still positive) and chosen the minuses such that the determinants reduce to determinants of positive definite matrices in known cases. We refer to this relation as the generalized Monge-Ampère equation. If the generalized Kähler potential K satisfies this non-linear PDE then K gives rise to a generalized Calabi-Yau metric structure. For the non-degeneracy of the metric it is important that the determinants in (3.14) are nowhere zero.

Using the relation (2.5) we can write down an explicit formula for the dilaton Φ . We need to calculate the determinant of the metric in our coordinates. Using the identities from appendix A it is lengthy but straightforward to calculate the determinant of the metric⁴

$$\sqrt{\det g_{\mu\nu}} = \frac{(-1)^{d_s d_c}}{\det K_{LR}} \det \begin{pmatrix} -K_{l\bar{l}} & -K_{lr} & -K_{l\bar{t}} \\ -K_{\bar{r}\bar{l}} & -K_{\bar{r}r} & -K_{\bar{r}\bar{t}} \\ -K_{t\bar{l}} & -K_{tr} & -K_{t\bar{t}} \end{pmatrix} \det \begin{pmatrix} K_{l\bar{r}} & K_{l\bar{l}} & K_{l\bar{c}} \\ K_{r\bar{r}} & K_{r\bar{l}} & K_{r\bar{c}} \\ K_{c\bar{r}} & K_{c\bar{l}} & K_{c\bar{c}} \end{pmatrix} . \quad (3.15)$$

⁴The calculation of the determinant of the metric results in an absolute square. When taking the square root of the determinant the sign has been determined so that the square root is positive when the generalized Monge-Ampère equation is satisfied.

It should be stressed that the determinant of the metric is not invariant under coordinate transformations; it of course transforms as a density. It is therefore important to emphasize that the formula above is given in coordinates defined by the superfields and that in the case with semichiral superfields these coordinates are not adapted to either of the complex structures. Using (2.5) and (3.15) we get an expression for the dilaton Φ

$$\begin{aligned} e^{2\Phi} &= (-1)^{d_s d_c} \frac{e^{-f(\phi)} e^{-\bar{f}(\bar{\phi})}}{\det K_{LR}} \det \begin{pmatrix} -K_{l\bar{l}} & -K_{lr} & -K_{l\bar{e}} \\ -K_{\bar{r}\bar{l}} & -K_{\bar{r}r} & -K_{\bar{r}\bar{e}} \\ -K_{t\bar{l}} & -K_{tr} & -K_{t\bar{e}} \end{pmatrix} \\ &= \frac{e^{-g(\chi)} e^{-\bar{g}(\bar{\chi})}}{\alpha \det K_{LR}} \det \begin{pmatrix} K_{l\bar{r}} & K_{l\bar{l}} & K_{l\bar{e}} \\ K_{r\bar{r}} & K_{r\bar{l}} & K_{r\bar{e}} \\ K_{c\bar{r}} & K_{c\bar{l}} & K_{c\bar{e}} \end{pmatrix}. \end{aligned} \quad (3.16)$$

An important issue is how these relations behave under coordinate changes. For instance, under diffeomorphisms that preserve our superspace structure (so that coordinates associated with a particular superfield mixes only with coordinates associated to superfields that satisfy the same superspace constraints), i.e. diffeomorphisms satisfying $\mathbf{X}'_l(\mathbf{X}_l, \phi, \chi)$, $\mathbf{X}'_r(\mathbf{X}_r, \phi, \bar{\chi})$, $\phi'(\phi)$, and $\chi'(\chi)$, it is possible to show that

$$\det \begin{pmatrix} -K_{l\bar{l}} & -K_{lr} & -K_{l\bar{e}} \\ -K_{\bar{r}\bar{l}} & -K_{\bar{r}r} & -K_{\bar{r}\bar{e}} \\ -K_{t\bar{l}} & -K_{tr} & -K_{t\bar{e}} \end{pmatrix} \quad (3.17)$$

transforms with a factor

$$\det \left(\frac{\partial \mathbf{X}'_l}{\partial \mathbf{X}_l} \right) \det \left(\frac{\partial \mathbf{X}'_r}{\partial \mathbf{X}_r} \right) \det \left(\frac{\partial \chi'}{\partial \chi} \right) \times c.c. \quad (3.18)$$

while

$$\det \begin{pmatrix} K_{l\bar{r}} & K_{l\bar{l}} & K_{l\bar{e}} \\ K_{r\bar{r}} & K_{r\bar{l}} & K_{r\bar{e}} \\ K_{c\bar{r}} & K_{c\bar{l}} & K_{c\bar{e}} \end{pmatrix} \quad (3.19)$$

transforms with a factor

$$\det \left(\frac{\partial \mathbf{X}'_l}{\partial \mathbf{X}_l} \right) \det \left(\frac{\partial \mathbf{X}'_r}{\partial \mathbf{X}_r} \right) \det \left(\frac{\partial \phi'}{\partial \phi} \right) \times c.c. \quad (3.20)$$

In particular this means that the generalized Monge-Ampère equation (3.14) is invariant under these changes of coordinates if we take into account how $e^{f(\phi)}$ and $e^{g(\chi)}$ transform,

see (3.8). We can also investigate how the dilaton changes under coordinate transformations. Since $\det K_{LR}$ transforms with a factor

$$\det \left(\frac{\partial \mathbf{X}'_l}{\partial \mathbf{X}_l} \right) \det \left(\frac{\partial \mathbf{X}'_r}{\partial \mathbf{X}_r} \right) \times c.c. \quad (3.21)$$

and using the transformation of (3.20) we see that the dilaton does not change. Again we should use the correct transformations (3.8) for $e^{f(\phi)}$ and $e^{g(\mathbf{x})}$. Indeed this is what we expect since the dilaton Φ is a function.

It follows from (3.8) that we can always change to a coordinate system such that $f = g = 1$ in a given coordinate patch. We do this in what follows, simplifying the generalized Monge-Ampère equation (3.14). Furthermore, for the case $d_c + d_t \neq 0$ we can always choose the constant α to be 1. However, in the case $d_c = d_t = 0$ we cannot remove the constant α . So in the generic case where $d_c + d_t \neq 0$ the generalized Monge-Ampère equation becomes

$$(-1)^{d_s d_c} \det \begin{pmatrix} -K_{l\bar{l}} & -K_{lr} & -K_{l\bar{t}} \\ -K_{\bar{r}\bar{l}} & -K_{\bar{r}r} & -K_{\bar{r}\bar{t}} \\ -K_{t\bar{l}} & -K_{tr} & -K_{t\bar{t}} \end{pmatrix} = \det \begin{pmatrix} K_{l\bar{r}} & K_{l\bar{l}} & K_{l\bar{c}} \\ K_{r\bar{r}} & K_{r\bar{l}} & K_{r\bar{c}} \\ K_{c\bar{r}} & K_{c\bar{l}} & K_{c\bar{c}} \end{pmatrix}, \quad (3.22)$$

and the dilaton becomes

$$\begin{aligned} e^{2\Phi} &= (-1)^{d_s d_c} \frac{1}{\det K_{LR}} \det \begin{pmatrix} -K_{l\bar{l}} & -K_{lr} & -K_{l\bar{t}} \\ -K_{\bar{r}\bar{l}} & -K_{\bar{r}r} & -K_{\bar{r}\bar{t}} \\ -K_{t\bar{l}} & -K_{tr} & -K_{t\bar{t}} \end{pmatrix} \\ &= \frac{1}{\det K_{LR}} \det \begin{pmatrix} K_{l\bar{r}} & K_{l\bar{l}} & K_{l\bar{c}} \\ K_{r\bar{r}} & K_{r\bar{l}} & K_{r\bar{c}} \\ K_{c\bar{r}} & K_{c\bar{l}} & K_{c\bar{c}} \end{pmatrix}. \end{aligned} \quad (3.23)$$

which implies that for this choice of coordinates

$$e^{-4\Phi} \sqrt{\det g} = \det K_{LR}. \quad (3.24)$$

Apart from the issue of the covariance of (3.14), there are other ambiguities in finding the pure spinors. First of all, the relation between the form of the pure spinors ρ_1, ρ_2 and the bihermitian data is not one to one; there are ambiguities both in the definition of the pure spinors and in the Gualtieri map. In particular, an overall B -transform gives different but equivalent pure spinors. Also, our expression for the pure spinors uses only closed forms so the pure spinors themselves are trivially closed. The nontrivial condition comes from the normalization of the spinors. One could alternatively choose to include a normalization

factor in the definition of the pure spinors so that they would be automatically normalized to 1. Then they would however not be automatically closed and the nontrivial condition would come from imposing closedness. When all is taken into account, these ambiguities do not effect the final answer (3.14).

4 Special cases

In this section we consider a few special cases of the general equation (3.14). We will see that some special cases of (3.14) have already appeared in the literature.

4.1 Kähler case: $d_s = d_t = 0$

If $J_+ = J_-$ then $H = 0$ and the generalized Kähler manifold is a Kähler manifold. In this case $d_t = d_s = 0$ and the pure spinors (3.7) are

$$\rho_1 = d\phi^1 \wedge \dots \wedge d\phi^{d_c}, \quad (4.25)$$

$$\rho_2 = e^{-id(K_C J d\phi)}. \quad (4.26)$$

The equation (3.14) becomes

$$\det(K_{c\bar{c}}) = 1, \quad (4.27)$$

where we have absorbed α in a redefinition of K . This is the well-known complex Monge-Ampère equation implying Ricci flatness of the Kähler manifold. In this case the dilaton Φ is constant.

4.2 Symplectic case: $d_c = d_t = 0$

In the case when $d_c = d_t = 0$ both generalized complex structures \mathcal{J}_1 and \mathcal{J}_2 are of symplectic type. In this case the pure spinors become

$$\rho_1 = e^{-d(K_L dX_L) + id(K_L J dX_L - K_R J dX_R)}, \quad (4.28)$$

$$\rho_2 = e^{-d(K_R dX_R) - id(K_L J dX_L + K_R J dX_R)}. \quad (4.29)$$

The generalized Monge-Ampère equation (3.14) becomes

$$\det \begin{pmatrix} K_{l\bar{l}} & K_{lr} \\ K_{\bar{r}l} & K_{\bar{r}r} \end{pmatrix} = \alpha \det \begin{pmatrix} K_{l\bar{r}} & K_{l\bar{l}} \\ K_{r\bar{r}} & K_{r\bar{l}} \end{pmatrix}. \quad (4.30)$$

Here α cannot be removed neither by rescaling of K nor by a rescaling of coordinates and thus we are dealing with a family of equations.

To compute the dilaton in this case is straightforward. Using (3.16) we get the following expression for the dilaton Φ

$$\Phi = \frac{1}{2} \ln \frac{\det \begin{pmatrix} K_{l\bar{l}} & K_{lr} \\ K_{\bar{r}\bar{l}} & K_{\bar{r}r} \end{pmatrix}}{\det \begin{pmatrix} K_{lr} & K_{l\bar{r}} \\ K_{\bar{l}r} & K_{\bar{l}\bar{r}} \end{pmatrix}}. \quad (4.31)$$

For the special case $d_s = 1$ (4.30) collapses to

$$K_{rl}K_{\bar{r}\bar{l}} + \alpha K_{r\bar{l}}K_{\bar{r}l} = (1 + \alpha)K_{l\bar{l}}K_{r\bar{r}}, \quad (4.32)$$

which is equivalent to the statement that

$$J_+J_- + J_-J_+ = 2 \left(\frac{1 - \alpha}{1 + \alpha} \right) \mathbb{I} \quad (4.33)$$

and, as shown in [5], when $\left(\frac{1-\alpha}{1+\alpha}\right)^2 < 1$ (which is true since α is positive), it is possible to explicitly construct a hyperKähler structure so the geometry is hyperKähler. See [17] for further discussion. For general d_s , (4.33) becomes

$$J_+J_- + J_-J_+ = 2 \left(\frac{1 - \alpha^{\frac{1}{d_s}}}{1 + \alpha^{\frac{1}{d_s}}} \right) \mathbb{I} \quad (4.34)$$

which similarly implies that the geometry is hyperKähler and it automatically solves (4.30). We have checked that the expression (4.31) for the dilaton reduces to a constant when the geometry is hyperKähler.

4.3 $[J_+, J_-] = 0$: $d_s = 0$

Consider the case when $d_s = 0$ and thus $[J_+, J_-] = 0$. This manifold admits a local product structure. The pure spinors become

$$\rho_1 = d\phi^1 \wedge \dots \wedge d\phi^{d_c} e^{id(K_T J d\chi)}, \quad (4.35)$$

$$\rho_2 = d\chi^1 \wedge \dots \wedge d\chi^{d_t} e^{-id(K_C J d\phi)}. \quad (4.36)$$

The generalized Monge-Ampère equation (3.14) collapses to

$$\det(K_{c\bar{c}}) = \det(-K_{t\bar{t}}), \quad (4.37)$$

where α can be absorbed either in rescaling of K or in the rescaling of χ - direction. In this case we have

$$\sqrt{g} = \det(K_{c\bar{c}}) \det(-K_{t\bar{t}}) \quad (4.38)$$

and thus from (2.5) the dilaton is

$$\Phi = \frac{1}{2} \ln \det(K_{c\bar{c}}) . \quad (4.39)$$

This equation first appeared in [18] (see also [19] and [20]).

For the case $d_c = d_t = 1$ we get a linear equation which is equivalent to the requirement of $N = (4, 4)$ supersymmetry for the corresponding sigma model. Indeed the general conditions for $N = (4, 4)$ supersymmetry for models with $d_s = 0$ and $d_c = d_t$ [11] can be written in matrix form as follows

$$K_{c\bar{c}} + K_{t\bar{t}} = 0 \quad (4.40)$$

$$(K_{c\bar{c}})^T = \overline{K_{c\bar{c}}} \quad (4.41)$$

which certainly gives a solution of (4.37). In fact we need only condition (4.40) to solve the generalized Monge-Ampère equation (4.37). Thus we can produce many solutions of this non-linear PDE.

4.4 $d_t = 0$

Let us finally comment on the case when $d_t = 0$ and thus one of the generalized complex structure is of symplectic type. The corresponding pure spinors are

$$\rho_1 = d\phi^1 \wedge \dots \wedge d\phi^{d_c} e^{-d(K_L dX_L) + id(K_L JdX_L - K_R JdX_R)} , \quad (4.42)$$

$$\rho_2 = e^{-d(K_R dX_R) - id(K_C Jd\phi + K_L JdX_L + K_R JdX_R)} . \quad (4.43)$$

This case has been studied previously by Halmagyi and Tomasiello in [6], especially for $D = 6$ due to its relevance for supergravity. Their pure spinors differ by a b -transform from the solution presented here. However their final generalized Monge-Ampère equation is the same as ours for this special case.

The explicit form of the dilaton Φ is provided by (3.16).

5 Geometrical considerations

In this section we want to elaborate on a few geometrical points related to the generalized Monge-Ampère equation (3.14). We are not going to derive any new results, however it

is instructive to consider the classical differential geometry of our equations. Moreover it provides a strong consistency check of our formulas.

The existence of a generalized Calabi-Yau metric structure is equivalent to the existence of spinors ϵ^\pm on the generalized Calabi-Yau space satisfying the following supersymmetry equations

$$\nabla^{(\pm)} \epsilon^{(\pm)} = 0 , \quad (5.44)$$

$$(d\Phi \pm \frac{1}{2}H) \epsilon^{(\pm)} = 0 . \quad (5.45)$$

The equation (5.44) is the so called gravitino equation while equation (5.45) is the dilatino equation. Here $\nabla^{(\pm)}$ are the covariant derivatives with connections with torsion $\Gamma^{(\pm)} = \Gamma \pm \frac{1}{2}g^{-1}H$ where Γ is the usual Christoffel symbol, while (5.45) involves the Clifford action of the one-form $d\Phi$ and the 3-form H on the spinors. See [13] for further details of the notation and the relation of these equations to generalized Calabi-Yau structures.

On a generalized Kähler manifold the equation (5.44) implies that the holonomies of the connections $\nabla^{(\pm)}$ are both in $SU(n)$ and the equation (5.45) implies a relation between the dilaton Φ and the rest of the geometrical data (g, H, J_\pm) , which is discussed below. Now we want to impose these conditions on the generalized Kähler geometry.

In the following we use formulas from [19],[21] and [22]. We define the $U(1)$ parts of the connections $\Gamma_\mu^{(\pm)}$

$$\Gamma_\rho^{(\pm)} = J_{\pm\nu}^\mu \Gamma_{\rho\mu}^{(\pm)\nu} , \quad (5.46)$$

the $U(1)$ parts of the curvatures

$$C_{\mu\nu}^{(\pm)} = J_{\pm\lambda}^\rho R_{\rho\mu\nu}^\lambda \quad (5.47)$$

and the J -trace of H

$$v_\rho^{(\pm)} = \pm J_{\pm\rho}^\mu H_{\mu\nu\sigma} J_\pm^{\nu\sigma} = 2J_{\pm\rho}^\mu \nabla_\nu J_\pm^\nu . \quad (5.48)$$

For formulae involving containing $v^{(+)}$ or $\Gamma^{(+)}$ it is useful to use complex coordinates adapted to J_+ while for formulae involving $v^{(-)}$ or $\Gamma^{(-)}$ we use complex coordinates adapted to J_- . In both cases, we use the indices a, b, \dots to label holomorphic coordinates and indices \bar{a}, \bar{b}, \dots to label anti-holomorphic coordinates. Then in such complex coordinate systems, we have

$$\Gamma_a^{(\pm)} = i \left(\Gamma_{ac}^{(\pm)c} - \Gamma_{a\bar{c}}^{(\pm)\bar{c}} \right) , \quad v_a^{(\pm)} = -\Gamma_{a\bar{c}}^{(\pm)\bar{c}} = \mp 2H_{a\bar{c}}^{(\pm)\bar{c}} , \quad (5.49)$$

where $H_{abc}^{(\pm)}$ is the $(2, 1)$ part of the three form H with respect to J_{\pm} , and

$$C_{\mu\nu}^{(\pm)} = \partial_{\mu}\Gamma_{\nu}^{(\pm)} - \partial_{\nu}\Gamma_{\mu}^{(\pm)} . \quad (5.50)$$

The $U(1)$ connection can be expressed in terms of the determinant of the metric and the one form $v^{(\pm)}$ as [21]

$$\Gamma_a^{(\pm)} = i \left(2v_a^{(\pm)} + \partial_a \ln \sqrt{\det g_{\mu\nu}} \right) \quad (5.51)$$

Equation (5.44) is equivalent to the holonomy of the $\nabla^{(\pm)}$ connections both being in $SU(n)$, i.e. $C_{\mu\nu}^{(\pm)} = 0$. At the same time, equation (5.45) implies that $v^{(\pm)} = -2d\Phi$ and thus Γ^{\pm} can be written as

$$\Gamma_a^{(\pm)} = i\partial_a \ln \left(e^{-4\Phi} \sqrt{\det g} \right) , \quad (5.52)$$

using the complex coordinates adapted to J_+ for $\Gamma^{(+)}$ and those adapted to J_- for $\Gamma^{(-)}$. Thus using these formulas we arrive at the statement that the equations (5.44) and (5.45) are equivalent the following conditions [19]

$$\partial\bar{\partial} \ln \left(e^{-4\Phi} \sqrt{\det g} \right) = 0 , \quad (5.53)$$

$$v^{\pm} = -2d\Phi , \quad (5.54)$$

where there are two equations of the form (5.53), one with J_+ complex coordinates and one with J_- complex coordinates. Then equation (5.53) implies that locally we can choose J_+ complex or J_- complex coordinates such that

$$e^{-4\Phi} \sqrt{\det g} = 1 . \quad (5.55)$$

The transformations relating the complex coordinates adapted to J_+ or J_- to our superspace-inspired coordinates (ϕ, χ, X_L, X_R) are given in Appendix A. Using these, the equations given by (5.55) in J_+ coordinates and by (5.55) in J_- coordinates both give the same equation in ϕ, χ, X_L, X_R coordinates, and that is

$$e^{-4\Phi} \sqrt{\det g} = \det K_{LR} . \quad (5.56)$$

and we have seen in (3.24) that there always exist coordinates where the above equation is identically satisfied⁵.

⁵This provides a proof that the generalized Monge-Ampère equation implies (5.44) and (5.45) or equivalently (5.53) and (5.54). We believe that the converse is also true, namely that the equations (5.53) and (5.54) imply our generalized Monge-Ampère equation. At present we have only been able to prove this for the case where $d_s = 0$ in which case there is a simple formula for $v^{(\pm)}$ in terms of the generalized Kähler potential.

Within the present framework it is easy to show that we are dealing with the solution of the equation (2.6). Let us work in J_+ complex coordinates. In this case the Ricci tensor can be written as follows [21]

$$R_{ab}^{(+)} = \nabla_a^{(-)} v_b^{(+)} , \quad (5.57)$$

$$R_{a\bar{b}}^{(+)} = \nabla_a^{(-)} v_{\bar{b}}^{(+)} - iC_{a\bar{b}}^{(+)} - 2(\partial_a v_{\bar{b}}^{(+)} - \partial_{\bar{b}} v_a^{(+)}). \quad (5.58)$$

Using equation (5.55) we find that the $U(1)$ connection $\Gamma_\mu^{(+)}$ is zero, so $C_{a\bar{b}}^{(+)} = 0$. Thus substituting $v^{(+)} = -2d\Phi$ into (5.57) and (6.70) we arrive at equation (2.6). Then the conditions for supersymmetry imply the equations of motion are satisfied, as was to be expected.

6 Physical interpretation

Generalized Kähler geometry is the target space geometry of classical $N = (2, 2)$ supersymmetric sigma models. In this section we comment on the physical interpretation of our generalized Monge-Ampère equation in relation to the finiteness, conformal invariance and supersymmetry of such sigma models.

A necessary condition for such a sigma model to be conformally invariant at one-loop is⁶

$$R_{\mu\nu}^{(+)} + 2\nabla_\mu^{(-)} \partial_\nu \Phi = 0 . \quad (6.59)$$

The condition for one-loop finiteness is however the weaker condition that [23]

$$R_{\mu\nu}^{(+)} + \nabla_\mu^{(-)} V_\nu + 2\partial_{[\mu} W_{\nu]} = 0 \quad (6.60)$$

for some V_μ, W_μ . This is sufficient to ensure that the one-loop counterterm is a total derivative when the classical equations of motion are used. Finally, we have seen that such a sigma model has target space supersymmetry if

$$C_{\mu\nu}^{(\pm)} = 0 , \quad (6.61)$$

$$v^{(\pm)} = -2d\Phi . \quad (6.62)$$

We now consider the implications of these for our geometries, using (5.57), (6.70) and

$$C_{ab}^{(\pm)} = 4i\partial_{[a} v_{b]}^{(\pm)} , \quad (6.63)$$

⁶This ensures that the dilaton beta-function is a constant at one loop. Full conformal invariance requires that this constant contribution is cancelled by contributions from ghosts and other matter fields.

which follows from (5.51). Consider first the condition (6.61) for $SU(n)$ holonomy, $C_{\mu\nu}^{(\pm)} = 0$. The condition (6.60) for one-loop finiteness was analysed in this case in [19] and is satisfied with $W_\mu = 2v_\mu^{(+)}$ and $V_\mu = -v_\mu^{(+)}$. From (6.63), $\partial_{[a}v_{b]}^{(\pm)} = 0$, so that locally

$$v_a^{(\pm)} = -2\partial_a(A^{(\pm)} + iB^{(\pm)}) \quad (6.64)$$

for some $A^{(\pm)}, B^{(\pm)}$. Then using (5.51) we have the Monge-Ampere equations [19]

$$\partial\bar{\partial}\ln\left(e^{-4(A^{(\pm)}+iB^{(\pm)})}\sqrt{\det g}\right) = 0. \quad (6.65)$$

For supersymmetry we need to impose in addition (6.62), which implies

$$A^{(\pm)} = \Phi, \quad dB^{(\pm)} = 0. \quad (6.66)$$

We then recover our Monge-Ampere equations [19]

$$\partial\bar{\partial}\ln\left(e^{-4\Phi}\sqrt{\det g}\right) = 0 \quad (6.67)$$

and find that the condition (6.59) for conformal invariance is satisfied. Thus the $SU(n)$ holonomy condition (6.61) implies one-loop finiteness, and the second supersymmetry condition (6.62) implies one-loop conformal invariance, so that supersymmetric backgrounds are conformal.

Next, we assume the conformal condition (6.59). Defining

$$k_a^{(\pm)} = v_a^{(\pm)} + 2\partial_a\Phi \quad (6.68)$$

we find that

$$\nabla_a^{(\mp)}k_b^{(\pm)} = 0 \quad (6.69)$$

and

$$iC_{ab}^{(\pm)} = \nabla_a^{(\mp)}k_b^{(\pm)} - 2(\partial_ak_b^{(\pm)} - \partial_bk_a^{(\pm)}). \quad (6.70)$$

A Killing vector \mathcal{K}^μ preserves H provided $\mathcal{K}^\mu H_{\mu\nu\rho}$ is a closed 2-form. Then if $C_{ab}^{(\pm)} = 0$, we see that $k^{(\pm)\mu}$ are Killing vectors preserving H . Further, if in addition $C_{ab}^{(\pm)} = 0$, the Killing vectors can be written locally in terms of potentials

$$k_a^{(\pm)} = -2\partial_a(\Phi + A^{(\pm)} + iB^{(\pm)}). \quad (6.71)$$

A nice example⁷ of this situation would be $S^3 \times S^1$ which admits two different solutions of (6.59). For a given Kähler potential [24] we can identify two solutions for this geometry

⁷Our discussion is local and we do not take into account global issues, in particular those related to no-go theorems in supergravity for compactifications with fluxes.

$(g_{\mu\nu}, H_{\mu\nu\rho}, 0)$ and $(g_{\mu\nu}, H_{\mu\nu\rho}, \Phi)$ with Φ given by (3.16). The metric and H -field are the same for both solutions, but one has zero dilaton and the other one has non-zero dilaton. This unusual situation occurs since the dilaton satisfies the following equation

$$\nabla_{\mu}^{(-)} \partial_{\nu} \Phi = 0 \quad (6.72)$$

so that $K^{\mu} = g^{\mu\nu} \partial_{\nu} \Phi$ is a Killing vector preserving H . The solution with non-zero dilaton provides a supersymmetric supergravity solution (at least locally), while the one with zero dilaton it will not. However both backgrounds solve (6.59). A large class of non-compact solutions with Killing vectors was given in [28].

As an alternative to the geometrical considerations, one can try to find the conditions for conformal invariance directly by performing a one-loop calculation in $N = (2, 2)$ superspace. However, in the presence of semi-chiral superfields, this is not so simple. The main problem is again that there is no good understanding of the dilaton in the general situation. Thus in [25] the authors only study a necessary condition for conformal invariance, ultra-violet finiteness at one loop.⁸ Comparing our results with [25] we can see that the solutions of the generalized Monge-Ampère equation (3.14) are finite, as they should be. However not every finite solution corresponds to the solution of (3.14).

In order to show this we need to compare our result with [25]. To this end we rewrite the generalized Monge-Ampère equation using the well known property of block matrices

$$\det \begin{pmatrix} A & B \\ C & P \end{pmatrix} = \det(P) \det(A - BP^{-1}C) , \quad (6.73)$$

where P is assumed to be invertible. Then (3.14) can be rewritten as

$$\begin{aligned} & \frac{\det(K_{c\bar{c}}) \det \begin{pmatrix} K_{l\bar{r}} - K_{l\bar{c}} K_{\bar{c}c}^{-1} K_{c\bar{r}} & K_{l\bar{l}} - K_{l\bar{c}} K_{\bar{c}c}^{-1} K_{c\bar{l}} \\ K_{r\bar{r}} - K_{r\bar{c}} K_{\bar{c}c}^{-1} K_{c\bar{r}} & K_{r\bar{l}} - K_{r\bar{c}} K_{\bar{c}c}^{-1} K_{c\bar{l}} \end{pmatrix}}{\det(-K_{t\bar{t}}) \det \begin{pmatrix} -K_{l\bar{l}} + K_{l\bar{t}} K_{t\bar{t}}^{-1} K_{t\bar{l}} & -K_{lr} + K_{l\bar{t}} K_{t\bar{t}}^{-1} K_{tr} \\ -K_{\bar{r}l} + K_{\bar{r}t} K_{t\bar{t}}^{-1} K_{t\bar{l}} & -K_{\bar{r}r} + K_{\bar{r}t} K_{t\bar{t}}^{-1} K_{tr} \end{pmatrix}} \\ & = \alpha e^{-f(\phi)} e^{-\bar{f}(\bar{\phi})} e^{g(x)} e^{\bar{g}(\bar{x})} , \end{aligned} \quad (6.74)$$

where we have to assume the invertibility of $K_{c\bar{c}}$ and $K_{t\bar{t}}$. Taking the logarithm of the left hand side of (6.74) we get exactly the $N = (2, 2)$ superspace counterterm calculated in [25].

⁸ The value of the work [25] is that it contains general expression for the counterterms of the $N = (2, 2)$ sigma model, while the precise relation between finiteness and conformality is not addressed, see [21] for further comments regarding this issue. We hope to be able to return to this question in a future publication

Whenever the generalized Monge-Ampère equation (3.14) is satisfied, the counterterm vanishes as the superspace integration gives zero, which then implies one-loop ultra-violet finiteness of the sigma model. However, the general condition for one-loop finiteness is less restrictive than this: the counterterm vanishes if the integrand is given by a generalized Kähler gauge transformation, i.e.

$$\begin{aligned}
& \frac{\det(K_{c\bar{c}}) \det \begin{pmatrix} K_{l\bar{r}} - K_{l\bar{c}} K_{\bar{c}c}^{-1} K_{c\bar{r}} & K_{l\bar{l}} - K_{l\bar{c}} K_{\bar{c}c}^{-1} K_{c\bar{l}} \\ K_{r\bar{r}} - K_{r\bar{c}} K_{\bar{c}c}^{-1} K_{c\bar{r}} & K_{r\bar{l}} - K_{r\bar{c}} K_{\bar{c}c}^{-1} K_{c\bar{l}} \end{pmatrix}}{\det(-K_{t\bar{t}}) \det \begin{pmatrix} -K_{l\bar{l}} + K_{l\bar{t}} K_{\bar{t}t}^{-1} K_{t\bar{l}} & -K_{l\bar{r}} + K_{l\bar{t}} K_{\bar{t}t}^{-1} K_{t\bar{r}} \\ -K_{r\bar{l}} + K_{r\bar{t}} K_{\bar{t}t}^{-1} K_{t\bar{l}} & -K_{r\bar{r}} + K_{r\bar{t}} K_{\bar{t}t}^{-1} K_{t\bar{r}} \end{pmatrix}} \\
& = F(\phi, \chi, \mathbf{X}_l) G(\phi, \bar{\chi}, \mathbf{X}_r) F(\phi, \chi, \mathbf{X}_l) G(\phi, \bar{\chi}, \mathbf{X}_r) .
\end{aligned} \tag{6.75}$$

Thus we see again that the generalized Monge-Ampère equation is more restrictive than just requiring one-loop ultra-violet finiteness. In section 3 we discussed how these determinants transform. In the finiteness condition (6.75) there is no way to set F and G to one by an appropriate coordinate transformation respecting the superfield structure.

7 Summary

In this paper we have analyzed the local conditions for the existence of generalized Calabi-Yau metric structures. In the neighborhood of a regular point the geometry is characterized by a single function K which is subject to a non-linear PDE, a generalization of the complex Monge-Ampère equation. In deriving this equation we found an explicit local expression for the pair of closed pure spinors that encode the geometry.

One would clearly want to find solutions of the generalized Monge-Ampère equation. Since we are dealing with a highly non-linear PDE, this task is very hard. However, in some cases the generalized Monge-Ampère equation can be replaced by a collection of linear PDE's, which are stronger but easier to solve. Namely, if the matrices on the left and right side of (3.14) are of the same size then we can require that they are related to each other by a similarity transformation, e.g., by some constant matrix. This will give us a collection of linear PDE's and their solution will automatically satisfy the condition (3.14). Another approach would be to use sigma model dualities which would allow one to map the non-linear PDE to a linear one, very much in analogy with [26, 27].

Another interesting physical aspect is to understand the proper relation between the conformal invariance of the $N = (2, 2)$ supersymmetric sigma model and target space supersymmetry. In particular it would be important to understand the proper status of the

generalized Monge-Ampère equation, since we know from the examples that it can accommodate not only supersymmetric supergravity solutions, but also non-supersymmetric but conformal solutions. The presence of semi-chiral superfields complicate the situation and we do not have anything to say about the matter at the moment.

Acknowledgement:

We thank Alessandro Tomasiello, Frederik Witt for discussions. We are grateful to the 2009 Simons Workshop where part of this work was carried out, for providing a stimulating atmosphere. The research of UL was supported by VR grant 621-2009-4066. The research of MR was supported in part by NSF grant no. PHY-06-53342. The research of R.v.U. was supported by the Simons Center for Geometry and Physics as well as the Czech ministry of education under contract No. MSM0021622409. The research of M.Z. was supported by VR-grant 621-2008-4273.

A Appendix: Useful formulae from GK geometry

In this appendix we review the relevant formulae for the local description of generalized Kähler geometry in terms of the generalized Kähler potential K . The generalized Kähler manifold is given by a bihermitian structure (g, J_+, J_-) with the following integrability conditions

$$d_+^c \omega_+ + d_-^c \omega_- = 0, \quad dd_\pm^c \omega_\pm = 0, \quad (\text{A.1})$$

where $\omega_\pm = gJ_\pm$ and d_\pm^c are $i(\bar{\partial} - \partial)$ operators associated with the complex structures J_\pm . The NS-form H is defined from here as follows

$$H = d_+^c \omega_+ = -d_-^c \omega_- , \quad dH = 0 . \quad (\text{A.2})$$

In the neighborhood of a regular point (i.e., in neighbourhoods in which the rank of $(J_+ \pm J_-)$ is constant) the geometry can be solved in terms of a single real function K . Using the notation introduced in section 3 the complex structures J_\pm are given by the following expressions

$$J_+ = \begin{pmatrix} J_s & 0 & 0 & 0 \\ K_{RL}^{-1} C_{LL} & K_{RL}^{-1} J_s K_{LR} & K_{RL}^{-1} C_{LC} & K_{RL}^{-1} C_{LT} \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & J_t \end{pmatrix} \quad (\text{A.3})$$

and

$$J_- = \begin{pmatrix} K_{LR}^{-1}J_s K_{RL} & K_{LR}^{-1}C_{RR} & K_{LR}^{-1}C_{RC} & K_{LR}^{-1}A_{RT} \\ 0 & J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & -J_t \end{pmatrix}, \quad (\text{A.4})$$

where J_s , J_c and J_t are canonical complex structures of size $2d_s$, $2d_c$ and $2d_t$ respectively. Here we have also introduced the matrices C and A which are given by the commutator or the anticommutator of the appropriate part of the hessian of K with the canonical complex structure J : $C_{\bullet\bullet} = JK_{\bullet\bullet} - K_{\bullet\bullet}J$, $A_{\bullet\bullet} = JK_{\bullet\bullet} + K_{\bullet\bullet}J$. The expressions (A.3) and (A.4) are given in superspace inspired coordinates X_L, X_R, ϕ, χ . In coordinates adapted to J_+ , J_+ itself is of course diagonal but J_- is complicated. The form (A.3) we get by doing a coordinate transformation from coordinates adapted to J_+ to the superspace inspired coordinates. Similarly, in coordinates adapted to J_- , J_- is diagonal and the expression (A.4) is the coordinate transformation of that simple form of J_- from coordinates adapted to J_- to superspace inspired coordinates, see [5] for details. For our analysis we need the explicit form of those coordinate transformation matrices. If we call the coordinate basis vectors of the J_+ adapted coordinates $dX_L, dY_L, d\phi, d\chi$, then the transformation matrix taking us from the superspace inspired coordinate basis to the J_+ adapted coordinate basis is given by

$$\begin{pmatrix} dX_L \\ dY_L \\ d\phi \\ d\chi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ K_{LL} & K_{LR} & K_{Lc} & K_{Lt} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dX_L \\ dX_R \\ d\phi \\ d\chi \end{pmatrix}, \quad (\text{A.5})$$

while if we call the coordinate basis vectors of the J_- adapted coordinates $dY_R, dX_R, d\phi, d\chi$, then the transformation matrix taking us from the superspace inspired coordinate basis to the J_- adapted coordinate basis is given by

$$\begin{pmatrix} dY_R \\ dX_R \\ d\phi \\ d\chi \end{pmatrix} = \begin{pmatrix} K_{RL} & K_{RR} & K_{Rc} & K_{Rt} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dX_L \\ dX_R \\ d\phi \\ d\chi \end{pmatrix}. \quad (\text{A.6})$$

In this work however, the only thing that we will need is the determinant of those transformations. In particular the determinant of the metric will transform as

$$\det g_{\mu\nu} \longrightarrow (\det K_{LR})^2 \det g_{\mu\nu}, \quad (\text{A.7})$$

when we pass from J_{\pm} -complex coordinates to our superfield coordinates.

The integrability condition (A.1) can be solved locally in terms of two closed non degenerate two-forms [16]

$$\mathcal{F}_+ = \frac{1}{2}(B_+ - g)J_+ = \frac{1}{2}d(-K_R JdX_R - K_C Jd\phi + K_T Jd\chi) , \quad (\text{A.8})$$

$$\mathcal{F}_- = \frac{1}{2}(B_- + g)J_- = \frac{1}{2}d(K_L JdX_L + K_C Jd\phi + K_T Jd\chi) , \quad (\text{A.9})$$

where B_{\pm} are $(2,0) + (0,2)$ -forms with respect to J_{\pm} and $H = dB_{\pm}$. Using these formulas we can work out explicit expressions for the metric $g_{\mu\nu}$ and NS-form $H_{\mu\nu\rho}$ and other objects.

For further detail on the local description of generalized Kähler geometry and its relation to supersymmetric sigma models the reader may consult [5, 14, 15, 16].

B Appendix: Proof

In this appendix we prove that the pure spinors given by (3.7) encode the generalized Kähler geometry. Let us start by a reminder of the relation between the bihermitian and the generalized complex descriptions of generalized Kähler geometry.

Theorem (Gualtieri's map [2]). *For the generalized Kähler geometry the corresponding generalized complex structures $\mathcal{J}_{1,2}$ can be reconstructed from the data (g, J_{\pm}, B)*

$$\mathcal{J}_{1,2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ (\omega_+ \mp \omega_-) & -(J_+^t \pm J_-^t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \quad (\text{B.1})$$

where $H = dB$.

Theorem. *The generalized complex structures $\mathcal{J}_{1,2}$ correspond to the closed pure spinors $\rho_{1,2}$ defined in (3.7) with B given below in (B.13) such that $H = dB$.*

Our goal is to show that the pure spinors annihilate the same maximally isotropic spaces as defined in terms of $\mathcal{J}_{1,2}$. Using Gualtieri's map we see that an eigenvector $X + \eta$ of \mathcal{J}_1 with eigenvalue $\pm i$ satisfies

$$\begin{aligned} J_+^t(g(X) + \eta + i_X B) &= \mp i(g(X) + \eta + i_X B) , \\ J_-^t(g(X) - \eta - i_X B) &= \mp i(g(X) - \eta - i_X B) \end{aligned} \quad (\text{B.2})$$

and an eigenvector $X + \eta$ of \mathcal{J}_2 with eigenvalue $\pm i$ satisfies

$$\begin{aligned} J_+^t(g(X) + \eta + i_X B) &= \mp i(g(X) + \eta + i_X B) , \\ J_-^t(g(X) - \eta - i_X B) &= \pm i(g(X) - \eta - i_X B) . \end{aligned} \quad (\text{B.3})$$

The generalized complex structures $\mathcal{J}_{1,2}$ give rise to the decomposition $(T \oplus T^*) \otimes \mathbb{C} = L_1^+ \oplus L_1^- \oplus \bar{L}_1^+ \oplus \bar{L}_1^-$, where L_1^+ is $+i$ -eigenbundle for both $\mathcal{J}_{1,2}$ and L_1^- is $+i$ -eigenbundle for \mathcal{J}_1 and $-i$ -eigenbundle for \mathcal{J}_2 . Using (B.2) and (B.3) and the fact that the metric is bihermitian we see that these sub-bundles are defined as follows

$$\begin{aligned} L_1^+ &= \{X + g(X) - i_X B \mid X \in T_+^{(1,0)}\} = \{X - i_X(B + i\omega_+) \mid X \in T_+^{(1,0)}\} , \\ L_1^- &= \{X - g(X) - i_X B \mid X \in T_-^{(1,0)}\} = \{X - i_X(B - i\omega_-) \mid X \in T_-^{(1,0)}\} , \\ \bar{L}_1^+ &= \{X + g(X) - i_X B \mid X \in T_+^{(0,1)}\} = \{X - i_X(B - i\omega_+) \mid X \in T_+^{(0,1)}\} , \\ \bar{L}_1^- &= \{X - g(X) - i_X B \mid X \in T_-^{(0,1)}\} = \{X - i_X(B + i\omega_-) \mid X \in T_-^{(0,1)}\} . \end{aligned}$$

The pure spinor ρ_1 should annihilate $\bar{L}_1 = \bar{L}_1^+ \oplus \bar{L}_1^-$, i.e. for any $X + \eta \in \bar{L}_1$

$$\iota_X \rho_1 + \eta \wedge \rho_1 = 0 \quad (\text{B.4})$$

and the pure spinor ρ_2 should annihilate $L_1^- \oplus \bar{L}_1^+$, i.e. for any $X + \eta \in L_1^- \oplus \bar{L}_1^+$

$$\iota_X \rho_2 + \eta \wedge \rho_2 = 0 . \quad (\text{B.5})$$

Using the concrete form of $\rho_{1,2}$ and the different eigenbundles we get the following relations:

$$X \in T_+^{(0,1)} , \quad i_X N_1 = 0 , \quad i_X(R_1 + iS_1 + i\omega_+ - B) \wedge N_1 = 0 , \quad (\text{B.6})$$

$$X \in T_-^{(0,1)} , \quad i_X N_1 = 0 , \quad i_X(R_1 + iS_1 - i\omega_- - B) \wedge N_1 = 0 , \quad (\text{B.7})$$

$$X \in T_+^{(0,1)} , \quad i_X N_2 = 0 , \quad i_X(R_2 + iS_2 + i\omega_+ - B) \wedge N_2 = 0 , \quad (\text{B.8})$$

$$X \in T_-^{(1,0)} , \quad i_X N_2 = 0 , \quad i_X(R_2 + iS_2 + i\omega_- - B) \wedge N_2 = 0 , \quad (\text{B.9})$$

From appendix A we have the relations (A.8) and (A.9),

$$B_+ J_+ - \omega_+ = d(-K_R J dX_R - K_C J d\phi + K_T J d\chi) , \quad (\text{B.10})$$

$$B_- J_- + \omega_- = d(K_L J dX_L + K_C J d\phi + K_T J d\chi) , \quad (\text{B.11})$$

where $H = dB_+ = dB_-$ and B_\pm is a $(2,0) + (0,2)$ form with respect to J_\pm respectively. By direct computation one may show that

$$B_+ - B_- = d(K_T d\chi - K_C d\phi) \quad (\text{B.12})$$

and we can define B as follows

$$B = B_+ + d(K_C d\phi) = B_- + d(K_T d\chi) . \quad (\text{B.13})$$

Exactly this B appears in the statement of our theorem. Using these relations it is straightforward to prove the relations (B.6)-(B.9). We will illustrate the proof for (B.6) and the rest of the relations (B.7)-(B.9) are proven in the same way. For (B.6), the relation $i_X N_1 = 0$ is obviously satisfied since N_1 is holomorphic. Using the explicit definition of R_1 , S_1 and the properties (B.10) and (B.13) we get

$$\begin{aligned} i_X(R_1 + iS_1 + i\omega_+ - B) \wedge N_1 = \\ i_X[-d(K_L(1 - iJ)dX_L) - d(K_C(1 - iJ)d\phi)] \wedge N_1 , \end{aligned}$$

where we also used the fact that B_+ is $(2, 0) + (0, 2)$ -form with respect to J_+ and $X \in T_+^{(0,1)}$.

Finally $d(K_L(1 - iJ)dX_L)$ is a $(2, 0)$ -form⁹ with respect to J_+ and thus $i_X(d(K_L(1 - iJ)dX_L)) = 0$ and $d(K_C(1 - iJ)d\phi) \wedge N_1 = 0$ since N_1 already contains all of the $d\phi$. That ends the proof of (B.6). After similarly proving (B.7)-(B.9) we conclude that the pure spinors encode the correct information about the generalized Kähler geometry.

Moreover we can check that the pure spinors $\rho_{1,2}$ obey the correct relation with respect to the Mukai pairing, $(\rho_1, \rho_2) = (\rho_1, \bar{\rho}_2) = 0$. Then the only remaining nontrivial condition comes from $(\rho_1, \bar{\rho}_1) = \alpha(\rho_2, \bar{\rho}_2) \neq 0$ which we study in detail in section 3. The conditions $(\rho_1, \rho_2) = (\rho_1, \bar{\rho}_2) = 0$ can be checked by direct calculation. Calculating (ρ_1, ρ_2) we get

$$N_1 \wedge N_2 \wedge (R_1 - R_2 + i(S_1 - S_2))^{(2d_s + \lfloor \frac{d_c + d_t}{2} \rfloor)} , \quad (\text{B.14})$$

where $\lfloor \frac{d_c + d_t}{2} \rfloor$ indicates the integer part and

$$R_1 - R_2 + i(S_1 - S_2) = -d(K_C(1 - iJ)d\phi + K_T(1 - iJ)d\chi + 2K_L(1 - iJ)dX_L) . \quad (\text{B.15})$$

Because of the $N_1 \wedge N_2$ prefactor in (B.14), terms containing $d\phi$ or $d\chi$ in the parenthesis are projected out and (ρ_1, ρ_2) becomes equal to

$$N_1 \wedge N_2 \wedge (2d(K_L(1 - iJ)dX_L))^{(2d_s + \lfloor \frac{d_c + d_t}{2} \rfloor)} . \quad (\text{B.16})$$

The two-form $d(K_L(1 - iJ)dX_L)$ is of $(2, 0)$ -type. with respect to J_+ and the maximum nonzero power of this term is d_s . Thus we conclude that $(\rho_1, \rho_2) = 0$. Analogously we can prove $(\rho_1, \bar{\rho}_2) = 0$. The Mukai pairing $(\rho_1, \bar{\rho}_2)$ is

$$N_1 \wedge \bar{N}_2 \wedge (R_1 - R_2 + i(S_1 + S_2))^{(2d_s + \lfloor \frac{d_c + d_t}{2} \rfloor)} , \quad (\text{B.17})$$

⁹For the explanation of this fact see [5]. Otherwise it can be checked explicitly using the form of the complex structures.

where

$$R_1 - R_2 + i(S_1 + S_2) = d(K_C(1 - iJ)d\phi + K_T(1 + iJ)d\chi + 2K_R(1 - iJ)dX_R) . \quad (\text{B.18})$$

Thus $(\rho_1, \bar{\rho}_2)$ becomes equal to

$$N_1 \wedge \bar{N}_2 \wedge (2d(K_R(1 - iJ)dX_R)^{(2d_s + \lfloor \frac{d_c + d_t}{2} \rfloor)}) , \quad (\text{B.19})$$

and here two-form $d(K_R(1 - iJ)dX_R)$ is of $(2, 0)$ -type with respect to J_- and therefore (B.19) vanishes identically.

References

- [1] N. Hitchin, *Generalized Calabi-Yau manifolds*, Quart. J. Math. Oxford Ser. **54** (2003) 281-308 [arXiv:math.DG/0209099].
- [2] M. Gualtieri, *Generalized complex geometry*, Oxford University DPhil thesis, [arXiv:math.DG/0401221].
- [3] G. R. Cavalcanti, *New aspects of the dd^c -lemma*, Oxford University DPhil thesis, [arXiv:math/0501406].
- [4] F. Witt, *Special metric structures and closed forms*, Oxford University DPhil thesis, [arXiv:math/0502443].
- [5] U. Lindström, M. Roček, R. von Unge and M. Zabzine, *Generalized Kähler manifolds and off-shell supersymmetry*, Commun. Math. Phys. **269** (2007) 833 [arXiv:hep-th/0512164].
- [6] N. Halmagyi and A. Tomasiello, *Generalized Kähler Potentials from Supergravity*, Commun. Math. Phys. **291** (2009) 1 [arXiv:0708.1032 [hep-th]].
- [7] M. Grana, R. Minasian, M. Petrini and A. Tomasiello, *Supersymmetric backgrounds from generalized Calabi-Yau manifolds*, JHEP **0408** (2004) 046 [arXiv:hep-th/0406137].
- [8] C. Jeschek and F. Witt, *Generalised G_2 -structures and type IIB superstrings*, JHEP **0503** (2005) 053 [arXiv:hep-th/0412280].
- [9] M. Grana, R. Minasian, M. Petrini and A. Tomasiello, *Generalized structures of $N=1$ vacua*, JHEP **0511** (2005) 020 [arXiv:hep-th/0505212].

- [10] F. Witt, *Calabi-Yau manifolds with B-fields*, Rend. Sem. Mat. Univ. Politec. Torino **66** (2008) 1 [arXiv:0908.1941 [math.DG]].
- [11] S. J. Gates, C. M. Hull and M. Roček, *Twisted Multiplets And New Supersymmetric Nonlinear Sigma Models*, Nucl. Phys. B **248** (1984) 157.
- [12] M. Gualtieri, *Generalized complex geometry*, [arXiv:math.DG/0703298].
- [13] F. Witt, *Metric bundles of split signature and type II supergravity*, Recent developments in pseudo-Riemannian geometry, 455-494, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008. [arXiv:math/0610380].
- [14] U. Lindström, M. Roček, R. von Unge and M. Zabzine, *Linearizing Generalized Kähler Geometry*, JHEP **0704** (2007) 061 [arXiv:hep-th/0702126].
- [15] U. Lindström, M. Roček, R. von Unge and M. Zabzine, *A potential for generalized Kähler geometry*, to appear in *Handbook of pseudo-Riemannian Geometry and Supersymmetry*, IRMA Lectures in Mathematics and Theoretical Physics, arXiv:hep-th/0703111.
- [16] C. M. Hull, U. Lindström, M. Roček, R. von Unge and M. Zabzine, *Generalized Kahler geometry and gerbes*, JHEP **0910** (2009) 062 [arXiv:0811.3615 [hep-th]].
- [17] M. Göteman and U. Lindström, *Pseudo-hyperkahler Geometry and Generalized Kahler Geometry*, arXiv:0903.2376 [hep-th].
- [18] T. H. Buscher, *Quantum Corrections And Extended Supersymmetry In New Sigma Models*, Phys. Lett. B **159**, 127 (1985).
- [19] C. M. Hull, *Superstring Compactifications With Torsion And Space-Time Supersymmetry*, Turin Superunif. (1985) 347.
- [20] M. Roček, *Modified Calabi-Yau manifolds with torsion*, In *Yau, S.T. (ed.): Mirror symmetry I*, 421-429.
- [21] C. M. Hull, *Sigma Model Beta Functions And String Compactifications*, Nucl. Phys. B **267** (1986) 266.
- [22] C. M. Hull, *Compactifications of the Heterotic Superstring*, Phys. Lett. B **178** (1986) 357.

- [23] C. M. Hull and P. K. Townsend, *Finiteness And Conformal Invariance In Nonlinear Sigma Models*, Nucl. Phys. B **274** (1986) 349.
- [24] M. Roček, K. Schoutens and A. Sevrin, *Off-Shell WZW Models In Extended Superspace*, Phys. Lett. B **265** (1991) 303.
- [25] M. T. Grisaru, M. Massar, A. Sevrin and J. Troost, *The quantum geometry of $N = (2,2)$ non-linear sigma-models*, Phys. Lett. B **412** (1997) 53 [arXiv:hep-th/9706218].
- [26] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, *Hyperkahler Metrics and Supersymmetry*, Commun. Math. Phys. **108**, 535 (1987).
- [27] J. Bogaerts, A. Sevrin, S. van der Loo and S. Van Gils, *Properties of semi-chiral superfields*, Nucl. Phys. B **562** (1999) 277 [arXiv:hep-th/9905141].
- [28] E. Kiritsis, C. Kounnas and D. Lust, *A Large Class Of New Gravitational And Axionic Backgrounds For Four-Dimensional Superstrings*, Int. J. Mod. Phys. **A9**, 1361 (1994) [arXiv:hep-th/9308124].